Journal of Approximation Theory **94**, 73–106 (1998)
Article No. AT983179

On Some Classes of Polynomials Orthogonal on Arcs of the Unit Circle Connected with Symmetric Orthogonal Polynomials on an Interval

Alexei Zhedanov

Department of Physics, Donetsk University, Donetsk, 340055, Ukraine

Communicated by Alphonse P. Magnus

Received August 11, 1995; accepted in revised form June 4, 1997

Starting from the Delsarte–Genin (DG) mapping of the symmetric orthogonal polynomials on an interval (OPI) we construct a one-parameter family of polynomials orthogonal on the unit circle (OPC). The value of the parameter defines the arc on the circle where the weight function vanishes. Some explicit examples of OPC connected with generic Askey–Wilson polynomials are constructed. These polynomials can be considered, as a four-parameter extension of the Askey–Szegő polynomials on the unit circle. We also present examples of OPC having finite and infinite purely discrete spectrum and addition masses inside and outside the unit circle. Connections of DG transformation with sieved OPI and OPC, chain sequences, and the Bauer's numerical g-algorithm (in approximation theory) are analyzed. In particular, we construct some classes of "semiclassical" OPI and OPC starting from periodic solutions of the Bauer's g-algorithm. © 1998 Academic Press

1. INTRODUCTION

The monic polynomials orthogonal on the real axis are defined by the three-term recurrence relation [7; 30, Chap. III]

$$R_{n+1}(y)+b_nR_n(y)+v_nR_{n-1}(y)=yR_n(y), \quad R_0(y)=1, \quad R_1(y)=y-b_0, \tag{1.1}$$

where it is assumed that the recurrence parameters b_n, v_n are real and $v_n>0$. These conditions guarantee that the polynomials $R_n(y)$ are orthogonal on a (finite or infinite) interval of the real axis with some nondecreasing measure $d\mu(y)=W(y)\,dy$

$$\int_\alpha^\beta R_n(y)\,R_m(y)\,d\mu(y)=h_n\delta_{nm}, \tag{1.2}$$

0021-9045/98 $25.00

where the normalization constants are

$$h_n = u_1 u_2 \cdots u_n. \tag{1.3}$$

Symmetric orthogonal polynomials $S_n(x)$ on the interval (SOPI) are a special case of $R_n(x)$ when $b_n = 0$. For convenience we write down the recurrence relation for the polynomials $S_n(x)$ in somewhat different notations than for $R_n(y)$

$$S_{n+1}(x) + u_n S_{n-1}(x) = x S_n(x), \qquad S_0(x) = 1, \qquad S_1(x) = x. \tag{1.4}$$

Provided $u_n > 0$, the polynomials $S_n(x)$ are orthogonal on a *symmetric* interval $[-2c, 2c]$ of the real axis with some *even* weight function $w(x)$ (see, e.g.) [7, Chap. 1])

$$\int_{-2c}^{2c} S_n(x)\, S_m(x)\, d\mu(x) = h_n \delta_{nm}, \tag{1.5}$$

where the measure $\mu(x)$ is *odd* function, whereas the weight function $w(x)$ defined by $d\mu(x) = w(x)\, dx$ is even function $w(-x) = w(x)$. The polynomials orthogonal on the unit circle (OPC) $P_n(z)$ were introduced by G. Szegő [30, Chap. XI]. They can be defined by the recurrence relations

$$P_{n+1}(z) = z P_n(z) - a_n P_n^*(z) \tag{1.6}$$

and initial conditions

$$P_0(z) = P_0^*(z) = 1. \tag{1.7}$$

Here $P_n^*(z) = z^n \bar{P}_n(1/z)$ are so-called reciprocal polynomials and $a_n = -P_{n+1}(0)$ are reflection parameters. It is well known [14] that when the reflection parameters are arbitrary complex numbers such that $|a_n| < 1$ then the polynomials $P_n(z)$ are orthogonal on the unit circle with respect to some nondecreasing measure $\sigma(\theta)$

$$\int_0^{2\pi} P_n(e^{i\theta})\, P_m(e^{-i\theta})\, d\sigma(\theta) = \kappa_n \delta_{nm}, \tag{1.8}$$

where $\kappa_n = (1 - |a_0|^2)(1 - |a_1|^2) \cdots (1 - |a_{n-1}|^2)$.

If all the reflection parameters are real numbers satisfying the condition $-1 < a_n < 1$ then the measure satisfies the condition (up to an additive constant) [14] $\sigma(2\pi - \theta) = -\sigma(\theta)$. The weight function $\rho(\theta)$ defined by $d\sigma(\theta) = \rho(\theta)\, d\theta$ becomes then symmetric with respect to the real axis: $\rho(2\pi - \theta) = \rho(\theta)$. In this case all the polynomials $P_n(z)$ have only real coefficients in their power expansion, so $\bar{P}_n(z) = P_n(z)$. For this special case Szegő introduced

[30, f-la (11.5.2)] a mapping from the polynomials $R_n(y)$ on the *finite* interval onto the polynomials $P_n(z)$ on the unit circle. For this Szegő mapping Geronimus derived [14] explicit formulas for correspondence between recurrence coefficients v_n, b_n and reflection parameters a_n.

For the case of symmetric polynomials $S_n(x)$ on the interval the situation is simplified drastically and the corresponding mapping becomes much more simple than that proposed by Szegő. Explicitly this mapping was found by P. Delsarte and Y. Genin in 1986 [8–10]. We will call it Delsarte–Genin (DG) transformation. It is interesting to note that in implicit form DG transformation is contained in works by H. Wall dealing with transformations of the continued fractions of one type to another [31; 32, Chap. 15]. Namely, Wall obtained explicit transformation allowing us to map Stieltjes continued fraction on the interval onto the continued fraction defined on the unit circle. The Wall relation [31] between the Stieltjes parameters g_n for the continued fraction on the interval and Schur parameters γ_n for the continued fraction on the unit circle is nothing else than DG correspondence between recurrence parameters u_n and reflection parameters a_n. The reason is that continued fractions contain all information about orthogonal polynomials (see, e.g., [14] for a thorough study of the connection between continued fractions and polynomials orthogonal on the unit circle). Moreover this Wall relation was also exploited by Geronimus [14] in order to get some non-trivial results for the OPC.

It should be noted also that in fact the Szegő and DG mappings are equivalent, because of the well known correspondence between symmetric and non-symmetric orthogonal polynomials on the interval (see, e.g., [7, Chap. 1, Sect. 9]). We briefly consider this equivalence because as far as we know it was not mentioned in the literature.

In this paper we consider a slightly generalized DG transform with additional scaling parameter. This allows one to get a family of the OPC orthogonal on an arc of the unit circle. The length of this arc depends on the scaling parameter. Despite the fact that introducing such a scaling parameter seems to be a trivial procedure, it leads to some non-trivial conclusions concerning OPC. Namely it is shown that one can define a mapping from the polynomial orthogonal on a whole unit circle onto the polynomials orthogonal on an arc. The polynomials orthogonal on an arc of the unit circle are presently being intensively studied (see, e.g., [16, 26]) so we hope that our explicit examples will be useful for further investigations of such polynomials.

The paper is organized as follows. In Section 2 we reproduce basic results concerning Delsarte–Genin transformation with a scaling parameter. In Section 3 we derive a squeezing transformation allowing us to transform OPC (with real reflection parameters) living on a whole circle onto OPC living on an arc. In Section 4 the relations between so-called sieved orthogonal polynomials on the circle and on the interval are easily derived on the base of DG transformation. In Section 5 we propose some explicit classes of OPC connected with generic Askey–Wilson polynomials on the interval. In Section 6 another special classes of OPC are considered, in particular we construct examples of OPC having one or two limiting points of a (discrete) measure and OPC having discrete masses outside the unit circle. In Section 7 it is shown that well known and important Wall results concerning so-called chain sequences can be naturally described in terms of DG transformation. In particular, we consider spectral properties of corresponding OPC living on an arc. In Section 8 we show that a chain of successive DG transformations can be presented in a form which is equivalent to the Bauer's g-algorithm which is a powerful tool for numerical approximation problems. Some special (periodic) solutions of the g-algorithm are considered. Asymptotic behavior of the reflection parameters and the measures of corresponding OPC are analyzed.

2. DELSARTE–GENIN TRANSFORMATION WITH A SCALING PARAMETER

In this section we briefly recall the main statements concerning DG transformation with a slight generalization (taking into account the scaling parameter d). Most of these statements will be given without proofs because the proofs are almost the same as in [8–10].

Introduce the connection between the real variable x and complex variable z by the relation

$$x = d(z^{1/2} + z^{-1/2}), \tag{2.1}$$

where d is a real positive parameter and $z^{1/2}$ is understood as $z^{1/2} = r^{1/2} e^{i\theta/2}$ if $z = e^{i\theta}$ (i.e., we choose one branch of the two-valued function).

PROPOSITION 1. *The formula*

$$P_n(z) = \frac{d^{-n-1} z^{n/2} (z^{1/2} S_{n+1}(x) - S_{n+1}(2d)\, S_n(x)/S_n(2d))}{z-1} \tag{2.2}$$

defines the mapping from the symmetric polynomials $S_n(x)$ *on the interval satisfying the recurrence relation* (1.4) *onto the polynomials* $P_n(z)$ *obeying the recurrence relation* (1.6) *with the reflection parameters*

$$a_{n-1} = 1 - d^{-1} S_{n+1}(2d)/S_n(2d). \tag{2.3}$$

Note that from (2.2) it follows that $P_0(z)=1$ provided the initial conditions (1.4) for $S_n(x)$ are fulfilled. Moreover from (2.3) it is convenient *to define*

$$a_{-1}=-1, \tag{2.4}$$

for compatibility of the lhs and rhs of (2.3) when $n=0$.

PROPOSITION 2. *The inverse mapping from $P_n(z)$ to $S_n(x)$ has the form*

$$S_n(x)=\frac{d^n z^{-n/2}(P_n(z)+P_n^*(z))}{1-a_{n-1}} \tag{2.5}$$

$$u_n=d^2(1+a_{n-1})(1-a_{n-2}), \qquad n=1, 2, ... \tag{2.6}$$

Remark 1. For $n=1$ we have $u_1=2d^2(1+a_0)$ which is compatible with (2.4).

Remark 2. The mapping (2.5), (2.6) is used in the theory of discrete integrable systems as the so-called discrete Miura transformation from the Volterra chain to the discrete mKdV equation. For details see [22].

So far, the scaling parameter d was arbitrary. Now we choose d in such a way that the polynomials $P_n(z)$ will be orthogonal on the unit circle. Assume that $[-2c_0, 2c_0]$, $c_0>0$ is a finite *true interval of orthogonality* (TIO) for the polynomials $S_n(x)$. Recall that (in the terminology of [7, Chap. 1, Sect. 5]) the TIO is defined as the smallest closed interval containing all of the zeros of all $S_n(x)$. This means in particular that all the points of increase of the measure $\mu(x)$ are contained in the TIO.

PROPOSITION 3. *If the scaling parameter d is such that $d>c_0$, then for the reflection parameters one has the inequality $-1<a_n<1$.*

Proof. If d lies to the right of the true interval of orthogonality, then $2d>x_{nk}$ where the x_{nk} $(k=1, 2, ..., n)$ are zeros of the polynomials $S_n(x)$ for all $n=1, 2, ...$ Hence $S_n(2d)>0$ and from (2.3) we have that $a_n<1$. Using the recurrence relation (1.4) we can rewrite (2.3) in the form

$$a_{n-1}=-1+d^{-1}u_n S_{n-1}(2d)/S_n(2d), \tag{2.7}$$

whence $a_n>-1$.

Under such a choice of the scaling parameter d we see that the conditions for orthogonality of $P_n(z)$ are fulfilled and hence we have the

PROPOSITION 4. *Under the assumption of Proposition* 3, *the polynomials* $P_n(z)$ *are orthogonal on the unit circle with the measure* $\rho(\theta)$ *defined by the formula*

$$\rho(\theta) = \mu(x), \tag{2.8}$$

where the connection between x *and* θ *follows from* (2.1):

$$x = 2d \cos(\theta/2). \tag{2.9}$$

For the corresponding weight functions we have the relation

$$\rho(\theta) = \sin(\theta/2)\, w(x). \tag{2.10}$$

Note that as is seen from (2.9), the weight function $\rho(\theta)$ vanishes on some arc near $z = 1$

$$\rho(\theta) \equiv 0 \qquad \text{if} \quad \theta < \theta_0 \qquad \text{and} \qquad \theta > 2\pi - \theta_0 \tag{2.11}$$

(recall that $0 \leqslant \theta < 2\pi$), where θ_0 is defined by

$$\cos(\theta_0/2) = c/d. \tag{2.12}$$

In [8] the DG mapping (2.2) was considered for the special case $c = d = 1/2$. Introducing the scaling parameter d allows one to construct a one-parameter family of the polynomials $P_n(z; d)$ starting from the same polynomials $S_n(x)$ on the interval. The polynomials $P_n(z; d)$ differ from one another by the length of the arc where the weight function $\rho(\theta)$ vanishes. It is interesting to note that if $d = c_0$ we get "maximal" polynomials $P_n(z; c_0)$. For these "maximal" polynomials there is at least one point of increase in any small arc $-\varepsilon < \theta < \varepsilon$ of the unit circle for the corresponding measure $\mu(\theta)$.

Thus the class of "explicitly known" polynomials orthogonal on the unit circle (OPC) is wider than the class of "explicit" polynomials on the interval (OPI). However, if we wish to have the expression for the reflection parameters a_n in terms of elementary functions (in analogy with the case of "the most general classical" OPI) then only restricted values (if any) of the scaling parameter d should be appropriate.

Note also that H. Widom [33] obtained some interesting results for the OPC with weight function vanishing on the arc (2.11); however he used the more complicated procedure of mapping the *non-symmetric* OPI onto OPC proposed by G. Szegő [30, Sect. 11.5; 14]. Let us describe briefly, how the Szegő mapping for arbitrary non-symmetric OPI $R_n(y)$ (satisfying the recurrence relation (1.1)) can be obtained from (2.2). We assume that both parameters α and β (defining the ends of the orthogonality interval in (1.2)) are finite. Introduce so-called kernel polynomials [7, Chap. 1, Sect. 7] $R_n^*(y)$ by the formula

$$R_n^*(y) = \frac{R_{n+1}(y) - R_n(y)\, R_{n+1}(\alpha)/R_n(\alpha)}{y-\alpha}. \tag{2.13}$$

The polynomials $R_n^*(y)$ are orthogonal on the same interval $[\alpha, \beta]$ but with another weight function $W^*(y) = (y-\alpha)\, W(y)$ [7, Chap. 1, Sect. 7]. Corresponding recurrence coefficients v_n^* and b_n^* for the polynomials $R_n^*(y)$ are connected with v_n and b_n by means of the discrete Darboux transformation (we do not write these coefficients explicitly here; for details see [29]). Then, following Chihara's approach [7, Chap. 1, Sect. 9], we can construct *symmetric* OPI $S_n(x)$ by the formulas

$$S_{2n}(x) = R_n(y-\alpha), \qquad S_{2n+1}(x) = xR_n^*(y-\alpha), \tag{2.14}$$

where

$$x^2 = y - \alpha. \tag{2.15}$$

The polynomials $S_n(x)$ obey the recurrence relation (1.4) with the coefficients

$$u_{2n} = -v_n \frac{R_{n-1}(\alpha)}{R_n(\alpha)}, \qquad u_{2n+1} = -\frac{R_{n+1}(\alpha)}{R_n(\alpha)}. \tag{2.16}$$

These polynomials are orthogonal on the symmetric interval $[-2c, 2c]$, where $2c = \sqrt{\beta - \alpha}$ with the weight function

$$w(x) = 2\, |x|\, W(y). \tag{2.17}$$

Now we can apply the mapping (2.2) to the polynomials $S_n(x)$ in order to obtain OPC. In the special case $c = d = 1/2$, $\beta = -\alpha = 1$ the resulting mapping coincides with the Szegő mapping [30, Sect. 11.5; 14] from $R_n(y)$ to $P_n(z)$. Thus, the Szegő mapping is a consequence of the mapping (2.2). But, obviously, if we wish to construct explicit examples of OPC it is much more convenient to start directly from symmetric polynomials $S_n(x)$ than from non-symmetric ones.

Apart from (2.2) there exists one more mapping from symmetric OPI to a *two-parameter* family of OPC. In order to construct this mapping denote by s_n an *arbitrary* real solution of the recurrence relation (1.4) with spectral parameter $2d$:

$$s_{n+1} + u_n s_{n-1} = 2d s_n.$$

Obviously, one can present s_n in the form

$$s_n = Q_n(2d) + \beta S_n(2d), \tag{2.18}$$

where β is an arbitrary real parameter, and $Q_n(z)$ is a standard second-order function (i.e., another independent solution for the recurrence relation (1.4), see, e.g., [15, f-la (IV.1)]):

$$Q_n(z) = \int_{-2c}^{2c} \frac{S_n(x)\, d\mu(x)}{z-x}.$$

Then we can state the following

PROPOSITION 5. *The formula*

$$P_n(z; d, \beta) = d^{-n} z^{n/2} (S_n(x) - z^{-1/2} s_n S_{n-1}(x)/s_{n-1}) \tag{2.19}$$

defines a two-parameter family of the polynomials $P_n(z; d, \beta)$ satisfying the recurrence relation (1.6) *with the reflection parameters*

$$1 + a_n = d^{-1} s_{n+1}/s_n, \tag{2.20}$$

where the relation between the arguments x and z is the same as in (2.1).

PROPOSITION 6. *The inverse mapping from $P_n(z; d, \beta)$ to $S_n(x)$ is*

$$S_n(x) = \frac{d^n z^{-n/2} (z P_n(z) - P_n^*(z))}{z-1}, \tag{2.21}$$

$$u_n = d^2 (1 - a_n)(1 + a_{n-1}). \tag{2.22}$$

The proof of these propositions is quite similar to that for Propositions 1 and 2. For the correspondence of the weight functions we have (instead of (2.10)) the formula (up to a normalization constant)

$$\rho(\theta; d, \beta) = w(x)/\sin(\theta/2) + 2\beta\delta(\theta). \tag{2.23}$$

So, in contrast to the mapping (2.2), for the mapping (2.19) we have an additional discrete mass on the circle located at $z = 1$. Note that the scaling parameter d defines the endpoints of the arc where $\rho(\theta) \equiv 0$, whereas the parameter β determines the value of the discrete mass at $\theta = 0$.

3. SQUEEZING TRANSFORMATION FOR THE POLYNOMIALS ON THE UNIT CIRCLE

In this section we show that by choosing two different values of the scaling parameter d_0 and d_1 in DG transformation one can obtain a nontrivial squeezing transformation of the circle polynomials onto themselves.

Indeed, let us start with symmetric polynomials $S_n(x)$ having recurrence coefficients u_n and the weight function $w(x)$. Assume that the true interval of orthogonality is $[-2c_0, 2c_0]$. Choose two scaling parameters $d_0 \geqslant c_0$ and $d_1 > d_0$. Then by (2.2) we can construct two systems of OPC $P_n(z)$ and $\tilde{P}_n(z)$ having the reflection parameters

$$1 - a_{n-1} = \frac{S_{n+1}(2d_0)}{d_0 S_n(2d_0)}, \tag{3.1}$$

$$1 - \tilde{a}_{n-1} = \frac{S_{n+1}(2d_1)}{d_1 S_n(2d_1)}. \tag{3.2}$$

The weight functions for these two OPC are defined from the equations

$$\rho(\theta) = w(x)\sin(\theta/2), \qquad x = 2d_0\cos(\theta/2), \tag{3.3}$$

$$\tilde{\rho}(\theta) = w(y)\sin(\theta/2), \qquad y = 2d_1\cos(\theta/2). \tag{3.4}$$

Both weight functions live on the arcs $\theta_i < \theta < 2\pi - \theta_i$, $i = 0, 1$. Obviously the first arc ($i = 0$) coincides with a whole circle iff $d_0 = c_0$. The second arc always occupies only a part of the circle.

Now we can consider transformation from $P_n(z)$ to $\tilde{P}_n(z)$. Denote $\cosh\omega = d_1/d_0 > 1$. Using formulas (2.3) we arrive at the

PROPOSITION 7. *Let $P_n(z)$ be a set of OPC with reflection parameters satisfying $-1 < a_n < 1$ and having the weight function*

$$\rho(\theta) = f(\cos(\theta/2)), \tag{3.5}$$

defined on the arc $\theta_0 < \theta < 2\pi - \theta_0$, where $f(x)$ is some even function. Then the new OPC with the reflection parameters $\tilde{a}_n$ defined from

$$1 - \tilde{a}_{n-1} = \frac{1 - a_{n-1}}{\cosh\omega} \frac{e^{\omega}P_n(e^{2\omega}) + e^{-\omega}P_n^*(e^{2\omega})}{P_n(e^{2\omega}) + P_n^*(e^{2\omega})}, \qquad \omega > 0 \tag{3.6}$$

are orthogonal on the arc $\theta_1 < \theta < 2\pi - \theta_1$ with the weight function

$$\tilde{\rho}(\theta) = \frac{\sin(\theta/2)\, f(\cosh\omega\cos(\theta/2))}{\sqrt{1 - \cosh^2\omega\cos^2(\theta/2)}}, \tag{3.7}$$

where the boundary of the new arc is defined by

$$\cos(\theta_1/2) = \frac{\cos(\theta_0/2)}{\cosh \omega}. \tag{3.8}$$

Of course, explicit expression for the polynomials $\tilde{P}_n(z)$ in terms of $P_n(z)$ can be also easily found from (2.2) and (2.5); however, we will not write it down. In Proposition 7, ω is an arbitrary positive parameter. This parameter defines the degree of "squeezing" the arc of orthogonality of the OPC: when $\omega \to \infty$ the arc is collapsed to the point $\theta = \pi$ on the unit circle. When $\omega \to 0$ we return to initial polynomials $\lim_{\omega \to 0} \tilde{a}_n = a_n$ and obviously $\theta_1 \to \theta_0$. It should be noted that the weight function (3.7) is defined up to a normalization factor.

Note that the squeezing transformation can be interpreted in terms of the following commutative diagram

$$\begin{tikzcd} S_n(x) \arrow[r] \arrow[d, "d_0"] & \tilde{S}_n(x) \arrow[d, "d_0"] \\ P_n(z) \arrow[r] & \tilde{P}_n(z) \end{tikzcd} \tag{3.9}$$

where vertical arrows means DG transform (2.2) with *the same* scaling parameter d_0. The lower horizontal arrow means squeezing transform for the OPC. The upper horizontal arrow means trivial scaling transform for the symmetric OPI: $\tilde{S}_n(x) = \cosh^{-n} \omega S_a(\cosh \omega x)$, $\tilde{u}_n = u_n \cosh^{-2} \omega$.

Thus, in contrast to the trivial scaling transform for the OPI the squeezing transform for OPC is not trivial: it allows one to get OPC with the prescribed arc of orthogonality.

Consider a simple example showing that the squeezing transform can yield new types of OPC. Take the trivial case $a_n = 0$, $n = 0, 1, \ldots$. Then obviously $P_n(z) = z^n$, $P_n^*(z) = 1$. The region of orthogonality for these polynomials coincides with a whole circle ($\theta_0 = 0$) and the weight function is a constant $\rho(\theta) = 1$. Then choosing arbitrary ω we get from Proposition 7 the OPC with the reflection parameters

$$\tilde{a}_{n-1} = -\tanh \omega \tanh \omega n. \tag{3.10}$$

These reflection parameters correspond to OPC having the weight function

$$\tilde{\rho}(\theta) = \frac{\sin(\theta/2)}{\sqrt{1 - \cosh^2 \omega \cos^2(\theta/2)}}. \tag{3.11}$$

It is seen from (3.11) that the weight function indeed lives only on the arc $\theta_1 < \theta < 2\pi - \theta_1$ where

$$\cos(\theta_1/2) = \frac{1}{\cosh \omega}. \tag{3.12}$$

Note that the reflection parameters tend to $-\tanh \omega$ when $n \to \infty$. This example provides us with OPC having *asymptotically constant* reflection parameters. A generic class of such polynomials was studied in [13] and more recently in [16]. Geronimus showed that if $\lim_{n\to\infty} a_n = a$ such that $|a| < 1$ then the boundary θ_1 of the arc where the OPC live is defined just by the condition (3.12). So the squeezing transform may provide one with efficient tools for investigating asymptotics of corresponding OPC.

In this section we found a squeezing transformation only for OPC having *real* reflection parameters a_n (because DG mapping is valid only for such OPC). It would be interesting to construct a squeezing mapping for generic (complex) values of a_n.

4. SIEVED POLYNOMIALS ON THE CIRCLE AND ON THE INTERVAL

In this section we show how the DG mapping allows one to construct sieved polynomials on the interval starting from sieved polynomials on the unit circle.

The first explicit example of sieved polynomials on the unit circle was considered by Ya. Geronimus [14, f-la (24.11)]. General theory of such polynomials was proposed in [17].

Recall [17] that sieved polynomials the unit circle are defined by the recurrence relation (1.6) with "sieved" reflection parameters

$$a_n = \begin{cases} 0, & \text{if} \quad n \neq mk-1; \\ \alpha_{m-1}, & \text{if} \quad n = mk-1,\ m = 1, 2, ..., \end{cases} \tag{4.1}$$

where k is some fixed positive integer number (defining a "depth" of "sieve") and the parameters α_m are assumed to satisfy inequalities

$$-1 < \alpha_m < 1. \tag{4.2}$$

Let $\Phi_n(z)$ be the polynomials having the reflection parameters α_n. The condition (4.2) guarantees that the polynomials $\Phi_n(z)$ are orthogonal on the unit circle (1.8) with some weight function $\rho_\Phi(\theta)$. Then it is easily shown (see, e.g., [17]) that sieved polynomials $P_n(z)$ are expressed in terms of $\Phi_n(z)$ as

$$P_{nk+j}(z) = z^j \Phi_n(z^k), \qquad j = 0, 1, ..., k-1; \quad n = 0, 1, ... \tag{4.3}$$

whereas the weight function $\rho_P(z)$ for sieved polynomials $P_n(z)$ is expressed via $\rho_\Phi(\theta)$ as

$$\rho_P(\theta) = \rho_\Phi(k\theta). \tag{4.4}$$

Now we can construct corresponding symmetric polynomials $\tilde{S}_n(x)$ on the interval using mapping (2.5). The recurrence coefficients for these polynomials are determined from (2.6) and (4.1)

$$\tilde{u}_n = \begin{cases} d^2(1+\alpha_{m-1}), & \text{if} \quad n = mk; \\ d^2(1-\alpha_{m-1}), & \text{if} \quad n = mk+1, \\ d^2 & \text{for any other } n, \end{cases} \tag{4.5}$$

where d is an arbitrary positive parameter.

On the other hand we can introduce the symmetric polynomials $S_n(x)$ corresponding to the circle polynomials $\Phi_n(z)$ via the same mapping (2.5) but with another scaling parameter d_0. Then such a procedure induces a mapping from the symmetric polynomials $S_n(x)$ onto other symmetric polynomials $\tilde{S}_n(x)$ via the commutative diagram

$$\begin{array}{ccc} S_n(x) & \longrightarrow & \tilde{S}_n(x) \\ \uparrow & & \uparrow \\ \Phi_n(z) & \longrightarrow & P_n(z) \end{array} \tag{4.6}$$

where vertical arrows mean inverse DG transforms (2.5) and horizontal arrows mean transition to sieved polynomials on the circle and on the interval. The polynomials $\tilde{S}_n(x)$ are thus sieved polynomials with respect to $S_n(x)$. Using (2.3) we can rewrite expression (4.5) for the recurrence coefficients $\tilde{u}_n$ in terms of recurrence coefficients u_n for the "initial" polynomials $S_n(x)$

$$\tilde{u}_n = \begin{cases} d^2 u_m S_{m-1}(2d_0)/S_m(2d_0), & \text{if} \quad n = mk; \\ d^2 S_{m+1}(2d_0)/S_m(2d_0), & \text{if} \quad n = mk+1, \\ d^2 & \text{for any other } n. \end{cases} \tag{4.7}$$

The formulas (4.7) are nothing else than expressions for the recurrence coefficients of sieved symmetric OPI obtained by polynomial mapping in

[12] (where the formulas (4.7) are written a slightly different, but equivalent, form).

Obviously, the scaling parameter d is not essential (in contrast to the essential parameter d_0). Hence we can choose, e.g., $d=1$. Then from (2.8) we get the expression of the weight function $\tilde{w}(x)$ for sieved polynomials $\tilde{S}_n(x)$ in terms of the weight function $w(x)$ for the polynomials $S_n(x)$ (up to a normalization factor)

$$\tilde{w}(x) = U_{k-1}(x)\, w(d_0 T_k(x)), \tag{4.8}$$

where $T_k(x) = 2\cos(k\phi)$ and $U_k(x) = \sin((k+1)\,\phi)/\sin\phi$ are *monic* Chebyshev polynomials of the first and second kind ($x = 2\cos\phi$). The formula (4.8) also coincides with that obtained in [12] by another method.

Thus DG transformation yields a very natural method to construct sieved OPI. Note that the idea to use the procedure (4.6) to construct sieved OPI was proposed by Badkov [4] who applied, however, a much more complicated Szegő–Geronimus mapping from *non-symmetric* OPI onto the unit circle. Hence the formulas for the recurrence coefficients of sieved polynomials obtained in [4] are too complicated for practical usage. The simple formulas (4.7) were first obtained by Geronimo and Van Assche [12] without using DG transformation.

5. CIRCLE ANALOGUES OF THE ASKEY–WILSON POLYNOMIALS

In this section we consider an explicit example of new OPC connected with the Askey–Wilson polynomials (AWP) [3] $p_n(x; b_1, b_2, b_3, b_4; q)$ where x is the argument and the b_i are the parameters of the polynomials (the parameter q is chosen in standard manner: $0 < q < 1$).

We choose the symmetric AWP having the real parameters $b_1 = -b_2 = \alpha$, $b_3 = -b_4 = \beta$. Then these polynomials satisfy the recurrence relation (1.4) with

$$u_n = \frac{(1-q^n)(1-\alpha^2\beta^2 q^{n-2})(1+\alpha^2 q^{n-1})(1+\beta^2 q^{n-1})}{(1-\alpha^2\beta^2 q^{2n-3})(1-\alpha^2\beta^2 q^{2n-1})}. \tag{5.1}$$

If $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ then the measure $d\mu(x)$ for these polynomials is purely continuous and located on the interval $[-2, 2]$. If $\alpha > 1$ but $|\alpha\beta| < 1$ then, in addition, there are several discrete masses outside the interval $[-2, 2]$ located at the points

$$x_k = \pm(\alpha q^k + \alpha^{-1} q^{-k}), \tag{5.2}$$

where the number of points is determined from the condition $\alpha q^k > 1$.

Choose the scaling parameter to be

$$d = (\alpha + \alpha^{-1})/2. \tag{5.3}$$

Then using explicit formulas for the AWP [3] we easily find the reflection parameters for corresponding OPC

$$a_n + 1 = \alpha d^{-1}(1 - q^{n+1})(1 + \beta^2 q^n)(1 - \alpha^2 \beta^2 q^{2n+1})^{-1}. \tag{5.4}$$

Note that if $\alpha = 1$ then we get OPC introduced by R. Askey in [2] (so-called Askey–Szegő polynomials on the unit circle). The formula (5.4) provides us with the two-parameter extension of the Askey example.

The explicit expression for the measure on the unit circle for the OPC corresponding to (5.4) is found from the formula (2.8) and from explicit expression for the weight function of the symmetric Askey–Wilson polynomials [3]

$$w(x) = (4 - x^2)^{-1/2} \frac{h(x;\, 1)\, h(x;\, -1)\, h(x;\, q^{1/2})\, h(x;\, -q^{1/2})}{h(x;\, \alpha)\, h(x;\, -\alpha)\, h(x;\, \beta)\, h(x;\, -\beta)}, \tag{5.5}$$

where

$$h(x;\, a) = (ae^{i\phi}, ae^{-i\phi};\, q)_\infty, \qquad x = 2\cos\phi. \tag{5.6}$$

Note that only if $\alpha = 1$ (i.e., in the case of Askey–Szegő polynomials) we have $\phi = \theta$ and in this case the corresponding weight function has no lacuna on the unit circle. If $\alpha \neq 1$ then the weight function on the circle vanishes on the arc (2.11) where $\cos(\theta_0/2) = 2/(\alpha + \alpha^{-1})$. In particular, for $\beta = q^{1/2}$, $|\alpha| < 1$ we get q-ultraspherical polynomials [3, 21]. For their circular analogues we have a simple formula for the reflection parameters

$$a_n = \frac{(\alpha - \alpha^{-1})(1 + \alpha^2 q^{2n+2})}{(\alpha + \alpha^{-1})(1 - \alpha^2 q^{2n+2})}. \tag{5.7}$$

Hence, for $\alpha \neq 1$ the circular analogues of q-ultraspherical polynomials have a lacuna in the weight function located near $z = 1$.

Our second (more complicated) example is connected with the mapping of generic (four-parameter) AWP $p_n(y;\, b_1, b_2, b_3, b_4;\, q)$ onto the polynomials orthogonal on the unit circle. Recall that the monic AWP satisfy the three-term recurrence relation [3, 21]

$$p_{n+1} + \xi_n \eta_{n-1} p_{n-1} + (\xi_n + \eta_n - b_1 - b_1^{-1})\, p_n = y p_n, \tag{5.8}$$

where

$$\xi_n = \frac{b_1(1-q^n)(1-b_2 b_3 q^{n-1})(1-b_2 b_4 q^{n-1})(1-b_3 b_4 q^{n-1})}{(1-gq^{2n-2})(1-gq^{2n-1})} \tag{5.9}$$

$$\eta_n = \frac{b_1^{-1}(1-b_1 b_2 q^n)(1-b_1 b_3 q^n)(1-b_1 b_4 q^n)(1-gq^{n-1})}{(1-gq^{2n})(1-gq^{2n-1})} \tag{5.10}$$

and $g = b_1 b_2 b_3 b_4$.

The explicit expression for these polynomials is

$$p_n(y) = C_{n4}\Phi_3\left(\begin{matrix} q^{-n}, gq^{n-1}, b_1 e^{i\phi}, b_1 e^{-i\phi} \\ b_1 b_2, b_1 b_3, b_1 b_4 \end{matrix}; q; q\right), \tag{5.11}$$

where

$$C_n = (-1)^n \frac{b_1^{-n}(b_1 b_2; q)_n (b_1 b_3; q)_n (b_1 b_4; q)_n (gq^{-1}; q)_n}{(gq^{-1}; q^2)_n (g; q^2)_n} \tag{5.12}$$

and $y = -2\cos\phi$.

The continuous part of the measure for the polynomials $p_n(y)$ (on the interval $-2 < y < 2$) corresponds to the weight function

$$w(y) = (4-y^2)^{-1/2} \frac{h(y;1)\, h(y;-1)\, h(y;q^{1/2})\, h(y;-q^{1/2})}{h(y;b_1)\, h(y;b_2)\, h(y;b_3)\, h(y;b_4)} \tag{5.13}$$

(notations are the same as in (5.6)).

In order to construct OPC we should construct *symmetric extension* of the AWP using the procedure described above (see (2.14)).

For this goal, let us assume that $b_1 > 0$, $b_4 < 0$ and moreover $|b_k| < 1$ for all $k = 1, ..., 4$ (the latter condition guarantees that the weight function consists of continuous part only). Define the polynomials

$$\begin{aligned} S_{2n}(x) &= (-1)^n\, p_n(y; b_1, b_2, b_3, b_4; q) \\ S_{2n+1}(x) &= (-1)^n\, x p_n(y; qb_1, b_2, b_3, b_4; q), \end{aligned} \tag{5.14}$$

where the arguments of the polynomials are connected by the relation

$$x^2 = y + b_1 + b_1^{-1}. \tag{5.15}$$

One can verify that these polynomials are monic, symmetric, and satisfy the recurrence relation (1.4) with $u_{2n} = \xi_n$, $u_{2n+1} = \eta_n$ (initial conditions (1.7) are also valid). It is natural to call these polynomials a "symmetric extension" of the Askey–Wilson polynomials. They are orthogonal on *two distinct intervals* of the real axis: $|x|_{\min} < |x| < |x|_{\max}$ where $x^2_{\min} = b_1 + b_1^{-1} - 2$,

$x^2_{\max}=b_1+b_1^{-1}+2$. The weight function $w_S(x)$ for these "symmetrized" Askey–Wilson polynomials is

$$w_S(x)=2\,|x|\,w(y), \tag{5.16}$$

where $w(y)$ is the weight function (5.13) for the ordinary AWP.

Choose the scaling parameter

$$2d=\sqrt{b_1+b_1^{-1}-b_4-b_4^{-1}}>2. \tag{5.17}$$

Then, applying the mapping (2.2) (starting from the polynomials (5.14)) we find corresponding circle analogues of the four-parameter AWP. The reflection parameters can be found from (2.3):

$$\begin{aligned} 1-a_{2n}&=\frac{2b_1(1-b_2b_4q^n)(1-b_3b_4q^n)}{(b_1-b_4)(1-gq^{2n})} \\ 1-a_{2n+1}&=\frac{2(1-gq^n)(1-b_1b_4q^{n+1})}{(1-b_1b_4)(1-gq^{2n+1})}. \end{aligned} \tag{5.18}$$

It is interesting to note that in this case the weight function for the OPC vanishes on *two distinct arcs* on the unit circle. The first arc is near $z=1$ with endpoints defined by

$$\cos^2(\theta_0/2)=\frac{b_1+b_1^{-1}+2}{b_1+b_1^{-1}-b_4-b_4^{-1}}. \tag{5.19}$$

The second arc is around $z=-1$, $|\theta-\pi|<\theta_1$, where

$$\sin^2(\theta_1/2)=\frac{b_1+b_1^{-1}-2}{b_1+b_1^{-1}-b_4-b_4^{-1}}. \tag{5.20}$$

If $b_1=-b_4=1$ then both arcs disappear ($\theta_0=\theta_1=0$) and we obtain some simplification for the corresponding OPC. The reflection parameters become

$$a_{2n}=-\frac{(b_2+b_3)\,q^n}{1+b_2b_3q^{2n}}, \qquad a_{2n+1}=-\frac{(b_2b_3+q)\,q^n}{1+b_2b_3q^{2n+1}}. \tag{5.21}$$

The connection between the arguments y and θ now is $y=2\cos\theta$, and for the weight function of the OPC we have

$$\rho(\theta)=\frac{(qe^{2i\theta}, qe^{-2i\theta}; q^2)_\infty}{(b_2e^{i\theta}, b_2e^{-i\theta}, b_3e^{i\theta}, b_3e^{-i\theta}; q)_\infty}. \tag{5.22}$$

When $b_2=-b_3$ then $a_{2n}=0$ and hence we obtain sieved Askey–Szegő OPC ($k=2$ in the definition (4.1)). Then the weight function (5.22) is obtained from the weight function of the ordinary Askey–Szegő [2] by the change $\theta \to 2\theta$ (see (4.4)).

Thus we constructed four-parameter circle analogues of the AWP. Perhaps these explicit examples of the OPC are new. Note that *non-polynomial* (rational in argument) bi-orthogonal systems on the unit circle connected with AWP were considered in [1, 18, 19].

6. SOME SPECIAL CLASSES OF OPC

Consider some interesting special cases of the OPC related with limiting degenerating AWP. Our first example is connected with so-called symmetric Wall polynomials [7, Chap. 6, Sect. 11] having the recurrence coefficients

$$u_{2n} = bq^n(1-q^n), \qquad u_{2n+1} = q^{n+1}(1-bq^n). \tag{6.1}$$

When $0 < b < 1$ these polynomials are orthogonal on the interval with purely discrete measure concentrated at the points $x_k = \pm q^{(k+1)/2}$, $k = 0, 1, 2, \ldots$. Note that $x_\infty = 0$ is the accumulating point of the spectrum. It is easily seen that for the scaling parameter $2d = q^{1/2}$ corresponding OPC have the reflection parameters

$$a_{2n} = 1 - 2bq^n, \qquad a_{2n-1} = 1 - 2q^n. \tag{6.2}$$

Obviously, the corresponding weight function on the unit circle is also a purely discrete one with the mass concentrated at the points $\cos(\theta_k/2) = q^{k/2}$ with $z = -1$ being the accumulating point. This case is interesting as the first explicit example of OPC having an accumulating point on the unit circle. Note that for $b = q^{1/2}$ the Wall polynomials become discrete q-Hermite polynomials [7, Chap. 11, Sect. 10]. In this case the reflection parameters are very simple: $a_n = 1 - 2q^{(n+1)/2}$ (the discrete spectrum does not depend on the value of the parameter b).

Another example is connected with symmetrized Al–Salam–Carlitz polynomials (ASCP) with recurrence coefficients [7, Chap. 6, Sect. 10; 21]

$$u_{2n} = 1 - q^n, \qquad u_{2n+1} = bq^n. \tag{6.3}$$

When $b > 0$ the ASCP are orthogonal with purely discrete measure concentrated in the points

$$x_k = \pm\sqrt{1-q^k}, \qquad \pm\sqrt{1-bq^k}.$$

Choosing the scaling parameter $2d = \sqrt{1+b}$ we find that corresponding OPC have the reflection coefficients

$$a_{2n} = (b-1)/(b+1), \qquad a_{2n-1} = 1 - 2q^n.$$

Again the measure on the unit circle is purely a discrete one with *two* accumulating points $\theta_\infty = \pm 2\arccos(1/\sqrt{1+b})$. Note that when $b=1$ we get "sieved" discrete q-Hermite polynomials on the unit circle (i.e, $a_{2n}=0$).

Consider also a simple example of OPC having only a finite number of discrete masses on the unit circle. This example is connected with symmetric Krawtchouk polynomials. Omitting the details of identification we present the result. Corresponding reflection parameters are

$$a_n = (n+1-j)/j, \qquad n = 0, 1, 2, ..., 2j-1, \tag{6.4}$$

where $2j$ is some positive integer parameter. This case is a finite-dimensional one because $a_{2j-1} = 1$ (see, e.g., [14]). Hence we have only $2j$ discrete masses located at the points of the circle

$$\cos(\theta_k/2) = (k-j)/j, \qquad k = 1, 2, ..., 2j.$$

The values of these masses are

$$M_k = \frac{2^{-k}(2j)!}{k!(2j-k)!}, \qquad k = 1, 2, ..., 2j-1;$$

$$M_{2j} = 2^{1-2j} \qquad \text{(the latter mass is located at } \theta = 0\text{)}.$$

So far, we restricted ourselves with the case when the scaling parameter $2d$ lies beyond the orthogonality interval: $d > c_0$. What happens when $0 < d < c_0$? Then, as is seen from (2.3) and (2.7), at least for some values of n we will have the inequality $|a_n| > 1$. This indicates that the polynomials $P_n(z)$ may not be orthogonal on the unit circle. However, there are some simple cases when the measure for the polynomials $P_n(z)$ can be found explicitly. Consider, for example, the case when the polynomials $S_n(x)$ have such a spectrum that on some interval $[2c_1 < 2c_0]$ the measure consists of a *finite number* of discrete masses only (without continuous part), located at the points x_k inside this interval, where as usual, $2c_0$ is the endpoint of the true interval of orthogonality, and $c_1 > 0$. Choose the scaling parameter inside this interval: $c_1 < d < c_0$. Then one can easily show that the polynomials $P_n(z)$ (constructed by (2.2)) have a measure consisting of two parts: one part of the measure belongs to the unit circle, whereas the second part consists of a finite number of discrete masses located on the *positive points of the real axis* $z_k = \exp(\pm\omega_k)$ where

$$\cosh \omega_k = x_k/2d. \tag{6.5}$$

The points x_k in (6.5) belong to the interval $2d < x_k < 2c_0$. So, each point x_k from this interval generates two spectral points $z_k^+ = e^{\omega}$, $z_k^- = e^{-\omega}$ with equal masses. Another method to construct such measures was considered in [6] where the authors describe a procedure that allows adding a pair of masses (located inside and outside the unit circle) to the initial measure on the unit circle. In our approach these "additional" masses appear quite naturally from the mapping (2.2).

Consider a simple example. Take again the discrete q-Hermite polynomials with (slightly renormalized) recurrence coefficients $u_n = q^{n-1}(1-q^n)$. The spectrum is a purely discrete one located at the points $\pm q^k$, $k = 0, 1, \ldots$. Choose $2d_j = q^j$, $j = 0, 1, \ldots$. For $j = 0$ we return to the previously considered case when all the spectral points for $P_n(z)$ belong to the unit circle. If $j \geqslant 1$ then one can show that

$$S_n(q^j) = q^{n(n-1)/2} p_j(q^{-n}), \tag{6.6}$$

where $p_j(y)$ is a j-order polynomial of the argument y. These polynomials can be easily expressed in terms of q-hypergeometric function (see, e.g., [11]). Applying the mapping (2.2) we get the polynomials $P_n(z)$ with the reflection parameters

$$a_{n-1} = 1 - 2q^{n-j} \frac{p_j(q^{-n-1})}{p_j(q^{-n})}. \tag{6.7}$$

Clearly, the obtained polynomials $P_n(z)$ have infinite discrete spectrum on the unit circle located at the points $\theta_k = 2 \arccos(q^{k-j})$, $k = j, j+1, \ldots$, and a finite number of discrete masses inside and outside the unit circle located at the points $\exp(\pm\omega_k)$, $k = 0, 1, \ldots, j-1$ where ω_k is found from $\cosh(\omega_k/2) = q^{k-j}$.

As a simplest example consider the case $j = 1$. Then the reflection parameters have the explicit expression

$$a_{n-1} = 1 - 2q^{n-2} \frac{1 - q^n(1+q)}{1 - q^{n-1}(1+q)}. \tag{6.8}$$

Two additional masses are located at the points $e^{\pm\omega}$ where $\cosh(\omega/2) = q^{-1}$. Note that in this case the polynomials $P_n(z)$ are well defined only if $q^{-N} \neq 1 + q$. Otherwise, the coefficient a_N doesn't exist.

7. DG TRANSFORMATION AND CHAIN SEQUENCES

In this section we consider an interesting relationship of DG transformation with so-called chain sequences. Recall [7, Chap. 3; 32, Chap. IV, Sect. 19] that a sequence of positive numbers $\{u_n\}$, $n=1, 2, ...$, is called a chain sequence if there exists another sequence $\{g_n\}$, $n=0, 1, ...$, such that

$$u_n = g_n(1-g_{n-1}), \qquad n=1, 2, ... \tag{7.1}$$

with the additional requirement

$$0 \leqslant g_0 < 1, \qquad 0 < g_n < 1, \qquad n=1, 2, ... \tag{7.2}$$

It is obvious from (7.1) and (7.2) that the requirement $0<u_n<1$ is necessary for the sequence u_n to be a chain sequence; however, this requirement is not sufficient because, e.g., the constant sequence $u_n \equiv u > 1/4$ is not a chain sequence [32, Chap. IV, Sect. 19].

If $\{u_n\}$ is a chain sequence then the sequence $\{g_n\}$ is called a parameter sequence. In general, for the given $\{u_n\}$ there can exist many parameter sequences. The parameter sequence $\{m_n\}$ is called a *minimal parameter sequence* if $m_0=0$. The parameter sequence $\{M_n\}$ is called a *maximal parameter sequence* if for any other parameter sequence $\{g_n\}$ one has $g_n \leqslant M_n$. If the minimal and the maximal parameter sequences coincide $m_n = M_n$ then the parameter sequence is uniquely determined by the sequence $\{u_n\}$.

The theory of chain sequences was developed by Wall [32, Clap. IV, Sect. 19] as a useful tool in the theory of continued fractions. T. Chihara showed the extreme importance of chain sequences in theory of orthogonal polynomials [7, Chap. 4]. Although the main results concerning representation of minimal and maximal parameter sequences in terms of initial chain $\{u_n\}$ were found by Wall [32, Chap. IV, Sect. 19], we would like to present here another treatment of these questions on the base of DG mappings (2.2) and (2.19). This allows one to obtain some new results, e.g., explicit expression for generic parameter sequence g_n and properties of corresponding OPC.

Indeed, put $d=1/2$ and choose the sequence $\{u_n\}$ to be the sequence of the recurrence coefficients for the corresponding symmetric polynomials $S_n(x)$ which are assumed to be orthogonal on the symmetric interval $[-2c, 2c]$, $c \leqslant 1/2$. Then the formula (2.6) provides solution for minimal parameter sequence

$$m_n = \frac{1+a_{n-1}}{2} = 1 - \frac{S_{n+1}(1)}{S_n(1)} \tag{7.3}$$

because $S_1(1)=1$ and hence $m_0=0$.

The formula (2.22) provides a generic non-minimal solution for the parametric sequence

$$g_n(\beta) = (1 - a_n)/2 = 1 - \frac{\phi_{n+1}(\beta)}{\phi_n(\beta)}, \tag{7.4}$$

where

$$\phi_n(\beta) = Q_n(1) + \beta S_n(1). \tag{7.5}$$

It is easily seen that the requirement (7.2) is fulfilled if and only if $\beta \geqslant 0$.

In particular, for $\beta = 0$ we get the maximal parameter sequence. Indeed, H. Wall obtained the expression for the maximal parameter chain M_n in terms of the continued fraction [32, f-la (19.6)]

$$M_n = 1 - \cfrac{u_{n+1}}{1 - \cfrac{u_{n+2}}{1 - \cfrac{u_{n+3}}{1 - \cdots}}} \tag{7.6}$$

provided that the continued fraction (7.6) converges. On the other hand, the continued fraction expression of the ratio $Q_{n+1}(z)/Q_n(z)$ for symmetric OPI is known to be (see, e.g., [15, f-la (IV.5)])

$$Q_{n+1}/Q_n(z) = \cfrac{u_{n+1}}{z - \cfrac{u_{n+2}}{z - \cfrac{u_{n+3}}{z - \cdots}}} \tag{7.7}$$

where conditions of convergence of the continued fraction (7.7) are the same as for the Stieltjes transform $Q_0(z)$ defined by

$$Q_0(z) = \int_{-2c}^{2c} \frac{w(x)\,dx}{z - x} \tag{7.8}$$

and playing a crucial role in spectral theory of OPI [15].

Comparing formulas (7.7) for $z = 1$ and (7.6) we obtain that $M_n = 1 - Q_{n+1}(1)/Q_n(1)$ which corresponds to the formula (7.4) for $\beta = 0$.

Obviously the maximal and minimal parameter sequences coincide iff $M_0 = 0$. Taking into account the relation $Q_1(z) = zQ_0(z) - 1$ (which is valid for any symmetric OPI) we see that the condition $M_0 = 0$ is equivalent to the condition

$$\lim_{z \to 1} Q_0(z) = \infty. \tag{7.9}$$

Thus the sequence $\{u_n\}$ defines the parameter sequence g_n uniquely iff the Stieltjes transform $Q_0(z)$ diverges for $z=1$. As a consequence we have that in this case necessary $[-1, 1]$ should be the true interval of orthogonality for the polynomials $S_n(x)$ (because otherwise $w(x)=0$ for $|c_0|<|x|<1$ and hence $Q_0(1)$ is well defined). In particular, the sequence $u_1=1/2$, $u_2=u_3=\cdots=1/4$ defines the Chebyshev polynomials of the first kind $T_n(x)$ having the Stieltjes transform [15] $Q_0(z)=(z^2-1)^{-1/2}$ where the principal brunch of the analytical function is chosen. The condition (7.9) is fulfilled and hence the parameter sequence in this case is determined uniquely: $g_0=0$, $g_n=1/2$, $n=1, 2, \ldots$. A less trivial example is provided by the sequence $u_n=q^{n-1}(1-q^n)$, $0<q<1$. These recurrence coefficients correspond to already considered discrete q-Hermite polynomials. From our consideration we easily find the minimal parameter sequence $m_n=1-q^n$. On the other hand, we know that the spectrum of the q-Hermite polynomials is a purely discrete one and located at the points $x_k=\pm q^k$, $k=0, 1, \ldots$. This means that the Stieltjes transform $Q_0(z)$ has simple poles at $z=x_k$. Hence $Q_0(z)$ diverges at $z=1$ and we obtain that in this case $m_n=M_n$ and so the parameter sequence is determined uniquely. Obviously the same situation will occur for any sequence u_n such that the measure for corresponding OPI $S_n(x)$ has an isolated discrete mass at $x=1$.

Summarizing we can formulate the following

PROPOSITION 8. *The problem of finding a parameter sequence g_n for the given chain sequence u_n is equivalent to finding reflection parameters a_n for OPC obtained from $S_n(x)$ by DG transform* (2.1.9) *with $d=1/2$. All possible sets of parameter sequences (if any) are given by the formulas* (7.4), (7.5) *for $0\leqslant\beta\leqslant\infty$. The minimal parameter sequence corresponds to $\beta=\infty$; the maximal parameter sequence corresponds to $\beta=0$. The parameter sequence is defined uniquely iff the Stieltjes transform $Q_0(z)$ diverges at $z=1$.*

As instructive example to this proposition consider the case of *constant* positive sequence $u_n=u<1/4$.

Then it is easy to find corresponding OPI

$$S_n(x)=\frac{e^{\omega_1(n+1)}-e^{\omega_2(n+1)}}{e^{\omega_1}-e^{\omega_2}}. \tag{7.10}$$

where $e^{\omega_{1,2}}$ are the roots of the equation

$$e^{2\omega}-xe^{\omega}+u=0 \tag{7.11}$$

and $x = e^{\omega_1} + e^{\omega_2}$. The true interval of orthogonality for the polynomials $S_n(x)$ is $[-2\sqrt{u}, 2\sqrt{u}]$ and the weight function is

$$w(x) = \sqrt{4u - x^2}. \tag{7.12}$$

This means that the polynomials $S_n(x)$ are nothing else than scaled Chebyshev polynomials $\gamma^{-n} U_n(\gamma x)$ of the second kind. The second-kind functions are

$$Q_n(x) = e^{\omega_2 n - \omega_1}, \tag{7.13}$$

where we choose $x > 2\sqrt{u}$ and

$$e^{-\omega_1} = \frac{x - \sqrt{x^2 - 4u}}{2u}$$

in order to provide true asymptotic behavior of the Stieltjes transform [15]

$$Q_0(x) = e^{-\omega_1} \to 1/x, \qquad \text{when} \quad x \to \infty. \tag{7.14}$$

From (7.4) we get a general solution for the parameter sequence

$$g_n(\beta) = e^{\omega_1} \frac{(\sqrt{1-4u} - \beta u)\, q^n + \beta u}{(\sqrt{1-4u} - \beta u)\, q^n + \beta e^{2\omega_1}}, \tag{7.15}$$

where $(x = 1)$

$$q = e^{\omega_2 - \omega_1} = \frac{1 - \sqrt{1-4u}}{1 + \sqrt{1-4u}}. \tag{7.16}$$

The maximal parameter sequence occurs when $\beta = 0$:

$$M_n = e^{\omega_1} = \tfrac{1}{2}(1 + \sqrt{1-4u}). \tag{7.17}$$

This result coincides with that obtained by Wall [32] by another method.

The minimal parameter sequence m_n is obtained for $\beta = \infty$

$$m_n = \frac{e^{\omega_2}(1 - q^n)}{1 - q^{n+1}}. \tag{7.18}$$

The expression (7.18) also reproduces Wall's result [32] for the minimal parameter sequence for the constant sequence $u_n = u$.

It is interesting to consider OPC corresponding to these parameter sequences. For generic β we have OPC with reflection parameters

$$a_n = 1 - 2g_n(\beta), \tag{7.19}$$

where $g_n(\beta)$ is defined by (7.15). Because $|q|<1$ we see from (7.15) that $\lim_{n\to\infty} a_n=\sqrt{1-4u}$, i.e., we again deal with OPC whose reflection parameters asymptotically behave as constants. It is therefore expected that these polynomials live on an arc.

Indeed the continuous part of the weight function is found from (2.23)

$$\rho(\theta)=\frac{\sqrt{4u-\cos^2(\theta/2)}}{\sin(\theta/2)}. \tag{7.20}$$

This continuous part of spectrum is located on the arc

$$\theta_1<\theta<2\pi-\theta_1, \qquad \text{where} \quad \cos\theta_1=2\sqrt{u}. \tag{7.21}$$

Apart from the continuous part on the arc there is one discrete mass 2β located at $\theta=0$. When $\beta=0$ (i.e., for the maximal parameter sequence) we have $a_n=\text{const}$ and hence arrive at the so-called Akhieser–Geronimus OPC having constant reflection parameters. The weight function (7.20) coincides with that found by Geronimus [15, 16] for these polynomials (for the case of *real* reflection parameters there are no discrete masses in the spectrum of Akhieser–Geronimus polynomials). For the generic case $\beta\neq 0$ we have that corresponding OPC have the weight functions which differ from one another only by the value of additional discrete mass at $\theta=0$, i.e., the family of OPC with reflection parameters (7.19) can be considered as a natural extension of the Akhieser–Geronimus polynomials emerging from the problem of finding a general parameter sequence for a given constant sequence $u_n=u$. All these OPC live on the arc and have the same continuous part of the weight function.

The case of minimal parameter sequence m_n demands a separate consideration, because direct limiting process for the weight functions and reflection parameters is impossible for $\beta\to\infty$. Hence we should consider the formulas (7.3) and (2.2) from which we get the reflection parameters

$$1-a_{n-1}=2e^{\omega_1}\frac{1-q^{n+2}}{1-q^{n+1}} \tag{7.22}$$

and the weight function

$$\rho(\theta)=\sin(\theta/2)\sqrt{4u-\cos^2(\theta/2)}. \tag{7.23}$$

Again the OPC with reflection parameters (7.22) live on the arc (7.21) and the measure is continuous only.

Thus DG mapping provides a quite natural treatment of the questions connected with chain sequences.

Finally, we would like to note that in [23] some interesting applications of chain sequences to SOPI were considered. Due to DG transformation from SOPI to OPC one should expect that the results of [23] could be applied to the polynomials orthogonal on the unit circle.

8. BAUER g-ALGORITHM AND "SEMICLASSICAL" OPI AND OPC

In this section we show that so-called g-algorithm [5], being a powerful tool in numerical approximations, has intimate connections with the theory of symmetric OPI, OPC. On the base of this algorithm one can construct new classes of OPI and OPC—so called "semiclassical" orthogonal polynomials in discrete arguments living on exponential grids.

Consider again symmetric OPI $S_n(x)$ satisfying the recurrence relation (1.4) and orthogonal on the finite interval $[-2c, 2c]$ with some weight function $w(x)$. Choose some positive parameter d beyond the true interval of orthogonality (i.e., $d > c_0$) and consider new symmetric OPI $\tilde{S}_n(x)$ obtained from $S_n(x)$ by means of the Christoffel transform [30, Sect. 2.5]

$$\tilde{S}_n(x) = \frac{S_{n+2}(x) - S_{n+2}(2d)\, S_n(x)/S_n(2d)}{x^2 - 4d^2}. \tag{8.1}$$

It is easily seen that the new polynomials $\tilde{S}_n(x)$ are monic symmetric OPI having the weight function

$$\tilde{w}(x) = \frac{(4d^2 - x^2)\, w(x)}{4d^2 - u_1}. \tag{8.1}$$

In (8.2) both weight functions are equally normalized:

$$\int_{-2c}^{2c} w(x)\, dx = \int_{-2c}^{2c} \tilde{w}(x)\, dx. \tag{8.3}$$

Note that the true interval of orthogonality for the polynomials $\tilde{S}_n(x)$ remains the same $[-2c_0, 2c_0]$. The polynomials $\tilde{S}_n(x)$ satisfy the recurrence relation

$$\tilde{S}_{n+1}(x) + \tilde{u}_n \tilde{S}_{n-1}(x) = x\tilde{S}_n(x), \tag{8.4}$$

where the new recurrence coefficients $\tilde{u}_n$ are expressed via initial ones u_n as

$$\tilde{u}_n = u_n \frac{S_{n-1}(2d)\, S_{n+2}(2d)}{S_n(2d)\, S_{n+1}(2d)}. \tag{8.5}$$

One can show that $\tilde{u}_n > 0$ iff $u_n > 0$ and $d > c_0$.

Consider the mapping (2.2) from OPI $S_n(x)$ to OPC $P_n(z)$ with the scaling parameter d. Obviously we have

$$u_n = d^2(1 + a_{n-1})(1 - a_{n-2}). \tag{8.6}$$

For the coefficients $\tilde{u}_n$ we find from (8.5) and (2.3)

$$\tilde{u}_n = d^2(1 - a_n)(1 + a_{n-1}). \tag{8.7}$$

On the other hand we can present the new coefficients $\tilde{u}_n$ in the form (8.6)

$$\tilde{u}_n = \tilde{d}^2(1 + \tilde{a}_{n-1})(1 - \tilde{a}_{n-2}), \tag{8.8}$$

where $\tilde{d}$ is an arbitrary new scaling parameter ($\tilde{d} > c_0$) and the $\tilde{a}_n$ are reflection parameters for another OPC $\tilde{P}_n(z)$ obtained from $\tilde{S}_n(x)$ by the mapping (2.2) with the scaling parameter $\tilde{d}$. Comparing (8.7) and (8.8) we arrive at the relation

$$\tilde{d}^2(1 + \tilde{a}_{n-1})(1 - \tilde{a}_{n-2}) = d^2(1 - a_n)(1 + a_{n-1}). \tag{8.9}$$

Now we can choose a new scaling parameter $\tilde{\tilde{d}}$, new reflection parameters $\tilde{\tilde{a}}_n$, obtaining the relation

$$\tilde{\tilde{d}}^2(1 + \tilde{\tilde{a}}_{n-1})(1 - \tilde{\tilde{a}}_{n-2}) = \tilde{d}^2(1 - \tilde{a}_n)(1 + \tilde{a}_{n-1}). \tag{8.10}$$

This procedure can be repeated: it is convenient to introduce the auxiliary discrete variable $t = t_0, t_0 + 1, t_0 + 2, \ldots$ such that $d = d(t_0)$, $\tilde{d} = d(t_0 + 1)$, $\tilde{\tilde{d}} = d(t_0 + 2)$, etc. (and analogously for $a_n(t)$). Thus we obtain the following set of relations

$$d^2(t+1)(1 + a_{n-1}(t+1))(1 - a_{n-2}(t+1)) = d^2(t)(1 - a_n(t))(1 + a_{n-1}(t)). \tag{8.11}$$

These relations are fulfilled for any admissible values of $t = t_0 + k$, $k = 0, 1, \ldots$. Thus we get some set of nonlinear equations (8.11) for the reflection parameters $a_n(t)$ depending on two discrete variables n and t. On the other hand, these relations can be considered as a useful representation for the chain of Christoffel transforms for the symmetric OPI $S_n(x)$. Indeed, any admissible solution of the set of equations (8.11) generates two chains of polynomials $P_n(z; t)$ an,d $S_n(x; t)$ connected (for each value of t) with one another by means of DG mapping (2.2) (by admissible we mean such solution that $a_{-1} = -1$ and $-1 < a_n < 1$ for all n).

It is worth mentioning that the relations (8.11) are well known in the theory of numerical approximations as the so-called *g-algorithm* discovered by F. Bauer [5]. Indeed, Bauer's *g-rhombus rules* are written in the form

[5, f-las (23), (24)] (we slightly modify Bauer's original notation introducing dependence in discrete time variable t)

$$g_{2n-2}(t+1)(\xi(t+1)-g_{2n-1}(t+1))=g_{2n}(t)(\xi(t)-g_{2n-1}(t)) \tag{8.12}$$

$$g_{2n-1}(t+1)(1-g_{2n}(t+1))=g_{2n+1}(t)(1-g_{2n}(t)), \tag{8.13}$$

where the coefficients $g_n(t)$ come from the theory of continued fractions as g-decomposition of Stieltjes S-fraction [32] (these coefficients are also connected with those considered in the previous section chain sequences), and the numbers $\xi(t)$ determine poles of the corresponding continued fraction [5]. Note that the g-algorithm can be considered as a more simple and flexible modification of the well known qd-algorithm [5, 20].

Let us introduce a connection between Bauer's and our parameters by the formulas

$$g_{2n+1}(t)=\xi(t)(1-a_{2n}(t))/2 \tag{8.14}$$

$$g_{2n}(t)=(1-a_{2n-1}(t))/2 \tag{8.15}$$

$$\xi(t)=d^2(t). \tag{8.16}$$

Then the Bauer's g-rhombus rules (8.12) and (8.13) become equivalent to relations (8.11). Note also that the relations (8.11) (in a slightly modified form) appeared in [29] as equations describing so-called "discrete-time Volterra chain."

It is instructive to see how Bauer's g-algorithm works in terms of our equations (8.11). Choose a sequence of scaling parameters $d(t_0+k)$, $k=0, 1, ...,$ and a sequence of "initial" reflection parameters $a_n(t_0)$. We will assume that the "boundary conditions" $a_{-1}(t)=-1$ are fulfilled for all possible values of the discrete time $t=t_0+k$ (these conditions correspond to Bauer's boundary conditions [5] $g_0(t)=1$ as is seen from (8.15)). Then from (8.11) we find

$$a_0(t_0+1)=-1+\frac{d^2(t_0)}{d^2(t_0+1)}(1+a_0(t_0))(1-a_1(t_0)), \tag{8.17}$$

i.e., we find explicitly $a_0(t_0+1)$ in terms of $a_i(t_0)$. Then other reflection parameters $a_n(t_0+1)$, $n=1, 2, ...,$ are determined *uniquely* from (8.11). Thus we construct the sequence $a_n(t_0+1)$. This procedure can be repeated to get the sequences $a_n(t_0+2)$, $a_n(t_0+3)$, ... Hence starting from a given sequence of reflection parameters $a_n(t_0)$ we can construct (step-by-step) a whole family of new sequences of reflection parameters $a_n(t_0+k)$. Obviously, as a byproduct, we get new families of corresponding OPI and OPC connected with one another by DG transform (2.2).

It would be interesting to find explicit solutions $a_n(t)$ for the Bauer g-algorithm, for any such solution provides a useful example of rational approximation of the function defined by a series (for details see, e.g., [5, 20]). Here we present one such (perhaps new) solution containing 4 free parameters. This solution follows from already considered reflection parameters for the circle analogues of the Askey–Wilson polynomials (5.18). Namely, we arrive at the following

PROPOSITION 9. *The reflection parameters $a_n(t)$ defined by* (5.18) *satisfy the equations* (8.11) *for the g-algorithm, where the parameters b_1, b_2, b_3 do not depend on the time t, whereas $b_4(t) = b_4(0)\, q^t$ ($t = t_0 + k$, $k = 0, 1, ...,$ t_0 is arbitrary, the parameters $d(t)$ are defined by* (5.17)).

The proof of this proposition is elementary and consists in direct substitution of (5.18) into Eqs. (8.11). Note that the boundary condition $a_{-1}(t) = -1$ is fulfilled for this solution. Clearly, taking appropriate special and limiting cases for the solution (5.18) we obtain a great majority of solutions of the g-algorithm which may be useful in questions connected with rational approximations.

Is it possible to get something beyond the Askey scheme [21] from the g-algorithm? The answer is yes. Another solutions of the Bauer g-algorithm (8.11) can be obtained by imposing some reductions. This method appears to be sufficiently effective. As a simplest (but not trivial!) example consider *periodic* reduction, i.e., we demand that after N steps the reflection parameters (and hence corresponding OPC) return to their initial values

$$a_n(t+N) = a_n(t) \tag{8.18}$$

(but in general, $d(t+N) \neq d(t)$ in order to avoid trivial solutions). Then we get a *finite* set of N nonlinear difference equations

$$\begin{aligned} &d^2(t_0+k+1)(1+a_n(t_0+k+1))(1-a_{n-1}(t_0+k+1)) \\ &\quad = d^2(t_0+k)(1-a_{n+1}(t_0+k)(1+a_n(t_0+k)), \qquad k = 0, 1, ..., N-1 \end{aligned} \tag{8.19}$$

for N unknown sequences $a_n(t_0+k)$. In general, the system (8.19) cannot be solved in terms of known special functions. We can extract however some information concerning spectral properties of the corresponding polynomials $S_n(x)$ (or $(P_n(z))$.

Indeed, after N steps of Christoffel transform (8.1) we get for the weight function $w(x;\, t_0+N)$ the expression (see (8.2))

$$w(x;\, t_0+N)=\gamma w(x;\, t_0)\prod_{k=0}^{N-1}(4d^2(t_0+k)-x^2), \tag{8.20}$$

where $\gamma>0$ is a normalization constant. On the other hand after N steps we return to the initial reflection parameters $a_n(t_0)$, hence for the recurrence parameters $u_n(t)$ we will have

$$\begin{aligned} u_n(t_0+N) &= d^2(t_0+N)(1-a_{n-1}(t_0+N))(1-a_{n-2}(t_0+N)) \\ &= \frac{d^2(t_0+N)}{d^2(t_0)}\, u_n(t_0). \end{aligned} \tag{8.21}$$

This means that the recurrence coefficients $u_n(t_0)$ return to their initial values with some scaling factor. Hence for corresponding OPI $S_n(x)$ we will have

$$S_n(x;\, t_0+N)=q^n S_n(x/q;\, t_0), \tag{8.22}$$

where

$$q=\frac{d(t_0+N)}{d(t_0)}. \tag{8.23}$$

We assume that $d(t_0+N)<d(t_0)$, hence $0<q<1$. For the weight function of the polynomials $S_n(x)$ we obviously have

$$w(x;\, t_0+N)=q^{-1}w(x/q;\, t_0). \tag{8.24}$$

Comparing (8.20) and (8.24) we arrive at the following q-difference equation for the weight function

$$w(q^{-1}x;\, t_0)=q\gamma w(x;\, t_0)\prod_{k=0}^{N-1}(4d_k^2-x^2), \tag{8.25}$$

where we denote for simplicity $d_k=d(t_0+k)$. Consider only the case when

$$c_0<d_0<d_1<\cdots<d_{N-1} \tag{8.26}$$

and assume that there are no q-commensuration relations between d_i, i.e., we demand that

$$d_i\neq q^j d_k \tag{8.27}$$

for all possible values $j=0, 1, \ldots$ and for $i, k=0, 1, \ldots, N-1$.

Among possible solutions of the functional equation (8.25) we should choose those $w(x)$ having needed properties for the weight function of OPI. Namely we should demand that $w(x)=0$ for all values $|x|>c_0$. Then from (8.25) we find that there exist *maximal* values x_{N-1} such that

$$w(q^{-1}x_{N-1};t_0)=0. \tag{8.28}$$

This implies that

$$x_{N-1}=\pm 2d_{N-1}. \tag{8.29}$$

From the condition (8.27) we find that analogous relations should be valid for other maximal points

$$x_k=\pm 2d_k, \qquad k=0,1,...,N-1. \tag{8.30}$$

Then starting from the maximal points x_k we construct a purely discrete measure. The discrete masses are located at points of N geometric series $\pm q^j x_k$, $k=0,1,,...,N-1$, $j=0,1,...,\infty$. The values of the masses at these points are

$$M_j^{(k)}=\frac{\nu_k\beta^{-j}}{\prod_{i=0}^{N-1}(q^2d_k^2/d_i^2;q^2)_j}, \qquad k=0,...,N-1,\ j=0,...,\infty, \tag{8.31}$$

where $\beta=\gamma\prod_{i=1}^{N-1}(4d_i^2)$ and ν_k are appropriate normalization constants. In order for all the masses $M_j^{(k)}$ to be positive we should demand that

$$0<qd_k/d_i<1 \qquad \text{for all } i, \quad k=0,1,...,N-1. \tag{8.32}$$

Taking into account (8.26) we see that this requirement is equivalent to

$$q<d_0/d_{N-1}<q^{-1}. \tag{8.33}$$

Thus the weight function $w(x)$ has discrete spectrum with the only limiting point $x=0$. From (2.8) we see that corresponding OPC also have purely discrete spectrum on the unit circle with the only limiting point $\theta=\pi$. According to the Stieltjes theorem (see, e.g., [16]) we can conclude that in this case the reflection parameters a_n should obey one of the conditions

$$\text{either} \qquad \lim_{n\to\infty} a_n(t_0)=1, \qquad \text{or} \qquad \lim_{n\to\infty} a_n(t_0)=-1. \tag{8.34}$$

Hence we can extract asymptotic behavior of the reflection parameters a_n despite the fact that the solution of Eqs. (8.19) is unknown.

Remark 3. Strictly speaking in this case the condition $d_i>c_0$ is violated because $c_0=d_{N-1}$ and $d_i<d_{N-1}$. However, this is not dangerous because

positivity of the weight function $w(x; t)$ is preserved at each step of the g-algorithm. The reason is that the measure is purely discrete and we choose the scaling parameters $2d_i$, $i=0, 1, ..., N-1$ coinciding with location of the N last discrete masses.

From the difference equation (8.25) for the weight function $w(x)$ it is seen that corresponding OPI $S_n(x)$ belong to so-called semi-classical polynomials in discrete argument, (SCPDA) (for definition of SCPDA see, e.g., [24, 27]). SCPDA possess many useful properties similar to those for the classical OPI in discrete arguments (by "classical" we mean Askey–Wilson polynomials and their degenerations according to the Askey scheme [21]). For example, Magnus showed [24] that all SCPDA possess difference-difference relations connecting polynomials with shifted argument x. In our case this property follows immediately from the definition of the polynomials $S_n(x)$. Indeed from (8.1), closure condition (8.22), and recurrence relation (1.4) it is seen that the obtained polynomials satisfy the difference-difference relation

$$S_n(q^{-1}x) = A_n(x)\, S_n(x) + B_n(x)\, S_{n-1}(x), \tag{8.35}$$

where $A_n(x)$, $B_n(x)$ are some rational functions in argument x whose order doesn't depend on n. But the relation (8.35) is a special case of general Magnus' relation [24] defining semi-classical OPI in discrete argument. Thus indeed the polynomials $S_n(x)$ obtained under the periodic closure condition (8.22) are SCPDA. It is natural to call corresponding OPC $P_n(z)$ also "semi-classical" polynomials on the unit circle. The difference-difference relations for semiclassical OPC $P_n(z)$ also exist and can be derived from (8.35) and DG mapping (2.2). Obviously these relations will be more complicated than those for semiclassical OPI (8.35) and we will not write them here.

Consider two simple examples.

For $N=1$ we have only Eq. (8.19)

$$d_1^2(1-a_{n-1}) = d_0^2(1-a_{n+1}) \tag{8.36}$$

from which we get the general solution with the restriction $a_{-1} = -1$

$$a_{2n-1} = 1-2q^{2n}, \qquad a_{2n} = 1-2bq^{2n}, \tag{8.37}$$

where b is an arbitrary parameter and $q = d_1/d_0 < 1$. Thus we get reflection parameters a_n corresponding to already considered circle analogues of the Wall polynomials (6.2). In this case semi-classical polynomials become purely "classical" polynomials (in a sense that the reflection parameters a_n are expressed in terms of the elementary function). Note that for the Wall polynomials the discrete masses are located at only *one* geometric series $x_k = \pm q^{k+1}$ as expected for $N=1$.

For $N=2$ we have two equations from (8.19)

$$(1+b_n)(1-b_{n-1})=\chi_1(1+a_n)(1-a_{n+1}); \tag{8.38}$$

$$(1+a_n)(1-a_{n-1})=\chi_2(1+b_n)(1-b_{n+1}), \tag{8.39}$$

where for simplicity we denote $\chi_1=(d_0/d_1)^2$, $\chi_2=(d_1/d_2)^2$, and $a_n=a_n(t_0)$, $b_n=a_n(t_0+1)$.

Multiplying (8.38) and (8.39) we get the relation

$$1-b_n=\frac{(\xi_1+(-1)^n\,\xi_2)\,q^n}{1-a_n}. \tag{8.40}$$

Analogously shifting $n\to n+2$ in (8.38) and multiplying (8.38) and (8.39) we get another relation

$$1+b_n=(\eta_1+(-1)^n\,\eta_2)\,q^{-n}(1+a_n)(1-a_{n-1})(1-a_{n+1}). \tag{8.41}$$

In (8.40) and (8.41), $q^{-2}=\chi_1\chi_2$ and ξ_1, ξ_2, η_1, η_2 are arbitrary parameters.

Compatibility of (8.40) and (8.41) with (8.38) (or (8.39)) leads to the restrictions

$$\xi_1\eta_2=\xi_2\eta_1, \qquad \xi_1\eta_1-\xi_2\eta_2=q\chi_1. \tag{8.42}$$

Moreover from (8.40) at $n=-1$ we get one more restriction

$$\xi_1-\xi_2=4q. \tag{8.43}$$

Excluding b_n from (8.40) and (8.41) we arrive at the equation for the reflection parameters a_n

$$\begin{aligned}2(1-a_n)=q^n(\xi_1+(-1)^n\,\xi_2)+(\eta_1+(-1)^n\,\eta_2)&\\ \times q^{-n}(1-a_n^2)(1-a_{n-1})(1-a_{n+1}).&\end{aligned} \tag{8.44}$$

Equation (8.44) is a second-order nonlinear difference equation for unknowns a_n. Such nonlinear difference equations are similar to those appearing in the theory of the "ordinary" semiclassical polynomials in continuous argument. In fact, these nonlinear equations go back to Laguerre (see, e.g., [25] and references therein). Note that similar equations appeared in application of the spectral transformation method to the second-order linear difference equations [28]. In general it is impossible to find a closed form solution of Eq. (8.44). Nevertheless for the admissible set of $\{a_n\}$ (i.e., corresponding to well defined OPC we can find asymptotics when $n\to\infty$. Indeed we already know that the Stieltjes theorem predicts that $\lim_{n\to\infty}=a$, where $a = \pm 1$. Assuming that $a = 1$ we find from (8.44) that $a_n - 1$ is exponentially small at large n,

$$1 - a_n \sim \rho_n q^{n/2}, \tag{8.45}$$

where ρ_n is a 4-periodic sequence determined by the relation

$$\rho_{n-1} \rho_{n+1} = 1/(\eta_1 + (-1)^n \eta_2). \tag{8.46}$$

The sequence ρ_n depends on 2 free parameters (say ρ_1 and ρ_2) which should be connected with initial conditions (i.e., with the value a_0, because the value a_1 is then determined uniquely from Eq. (8.44)) for the g-algorithm. It is natural to conjecture that such exponential asymptotics are valid for the reflection parameters a_n in the general case for arbitrary N.

ACKNOWLEDGMENTS

The author is grateful to S. Kharchev, A. Mironov, V. Spiridonov, S. Suslov, and L. Vinet for discussions; to R. Askey, L. Golinskii, A. Magnus, P. Nevai, F. Peherstorfer, and M. Rahman for stimulating communications; and to the referees for valuable remarks. I am especially indebted to A. Magnus for bringing to my attention very interesting papers [24, 25] before publication. The author also thanks the Centre de Recherches Mathématiques of the Université de Montréal for its hospitality.

REFERENCES

1. W. A. Al-Salam and M. E. H. Ismail, A q-beta integral on the unit circle and some biorthogonal rational functions, *Proc. Amer. Math. Soc.* **121** (1995), 553–561.
2. R. Askey, Comments, *in* "Gabor Szegő: Collected Papers," Vol. 1, pp. 806–811, Birkhäuser, Basel, 1982.
3. R. Askey, and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* **54** (1985), 1–55.
4. V. M. Badkov, Systems of orthogonal polynomials explicitly represented by the Jacobi polynomials, *Math. Notes* **42** (1987), 858–863.
5. F. L. Bauer, The g-algorithm, *J. Soc. Indust. Appl. Math.* **8** (1960), 1–17.
6. A. Cachafeiro and F. Marcellan, Modifications of Toeplitz matrices: Jump functions, *Rocky Mountain J. Math.* **23** (1993), 521–531.
7. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.
8. P. Delsarte, and Y. Genin, The split Levinson algorithm, *IEEE Trans. Acoust. Speech Signal Process.* **34** (1986), 470–478.
9. P. Delsarte and Y. Genin, Tridiagonal approach to the algebraic environment of Toeplitz matrices I. Basic results, *SIAM J. Matrix Anal. Appl.* **12**, No. 2 (1991), 220–238.
10. P. Delsarte and Y. Genin, Tridiagonal approach to the algebraic environment of Toeplitz matrices. II. Zero and eigenvalue problems, *SIAM J. Matrix Anal. Appl.* **12**, No. 3 (1991), 432–448.

11. G. Gasper and M. Rahman, “Basic Hypergeometric Series,” Cambridge Univ. Press, Cambridge, 1990.
12. J. Geronimo and W. Van Assche, Orthogonal polynomials via polynomial mappings, *Trans. Amer. Math. Soc.* **308** (1988), 559–581.
13. Ya. L. Geronimus, On the character of the solution of the moment in the case of limit-periodic associated fractions, *Izv. Akad. Nauk SSSR Ser. Mat.* **5** (1941), 203–210.
14. Ya. L. Geronimus, Polynomials orthogonal on a circle and their applications, *in* “Amer. Math. Soc. Transl. Ser. 1,” Vol. 3, pp. 1–78, Amer. Math. Soc., Providence, 1962.
15. Ya. L. Geronimus, Orthogonal polynomials, *in* “Two Papers on Special Functions,” Amer. Math. Soc. Transl. Ser. 2, Vol. 108, pp. 37–130, Amer. Math. Soc., Providence, RI, 1977.
16. L. Golinskii, P. Nevai, and W. Van Assche, Perturbation of orthogonal polynomials on an arc of the unit circle, *J. Approx. Theory* **83** (1995), 391–422.
17. M. E. H. Ismail and Xin Li, On sieved orthogonal polynomials. IX. Orthogonality on the unit circle, *Pacific J. Math.* **153** (1992), 289–297.
18. M. E. H. Ismail and D. R. Masson, q-Hermite polynomials, biorthogonal rational functions and q-beta integrals, *Trans. Amer. Math. Soc.* **346** (1994), 63–116.
19. M. E. H. Ismail and M. Rahman, Ladder operators for Szegő polynomials and related biorthogonal rational functions, *Proc. Amer. Math. Soc.* **124** (1996), 2149–2159.
20. W. B. Jones and W. J. Thron, “Continued Fractions: Analytic Theory and Applications,” Encyclopedia of Mathematics and Its Applications, Vol. 11, Addison–Wesley, Reading, Ma, 1980.
21. R. Koekoek and R. F. Swarttouw, “The Askey-scheme of Hypergeometric Orthogonal Polynomials and Its q-Analogue,” Report 94-05 TU, Delft, 1994.
22 S. Karchev and A. Mironov, Integrable structures of unitary matrix models, *Internat. J. Modern Phys. A* **7** (1992), 4803–4824.
23. R. Lasser, Orthogonal polynomials and hupergroups. II. The symmetric case, *Trans. Amer. Math. Soc.* **341** (1994), 749–770.
24. A. Magnus, Special nonuniform lattice (snul) orthogonal polynomials on discrete dense sets of points, *J. Comput. Appl. Math.* **65** (1995), 253–365.
25. A. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, *J. Comput. Appl. Math.* **57** (1995), 215–237.
26. F. Peherstorfer and R. Steinbauer, Orthogonal polynomials on arcs of the unit circle, I, *J. Approx. Theory* **85** (1996), 140–184; II, **87** (1996), 60–102.
27. A. Ronveaux, Discrete semi-classical orthogonal polynomials, *J. Approx. Theory* **46** (1986), 403–407.
28. V. Spiridonov, L. Vinet, and A. Zhedanov, Difference Schrödinger operators with linear and exponential spectra, *Lett. Math. Phys.* **29** (1993), 63–73.
29. V. Spiridonov and A. Zhedanov, Discrete Darboux transformations, the discrete-time Toda lattice, and the Askey–Wilson polynomials, *Methods Appl. Anal.* **2** (1996), 369–398.
30. G. Szegő, “Orthogonal Polynomials,” 4th ed., Amer. Math. Soc., Providence, 1975.
31. H. S. Wall, Some recent developments in the theory of continued fraction, *Bull. Amer. Math. Soc.* **47** (1941), 405–423.
32. H. S. Wall, “Analytic Theory of Continued Fractions,” Chelsea, New York, 1973.
33. H. Widom, The strong limit theorem for circular arcs, *Indiana Univ. Math. J.* **21** (1971), 277–283.